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Lyapunov Exponents for Burgers' Equation

Alexei Kourbatov[1]

[1] Formerly with Moscow Institute of Physics and Technology

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Abstract

We establish the existence, uniqueness, and stability of the stationary solution of the one-dimensional viscous Burgers equation with the Dirichlet boundary conditions on a finite interval. We obtain explicit formulas for solutions and analytically determine the Lyapunov exponents characterizing the asymptotic behavior of arbitrary solutions approaching the stationary one.

Keywords: nonlinear PDE, Burgers equation, boundary value problem, Dirichlet boundary conditions, Lyapunov exponent.

AMS Subject Classification: Primary 35K55; Secondary 35B40, 35C05.

Introduction

Burgers' equation has the same nonlinearity form as the Navier-Stokes equations [1]. It is often used as a model equation in studying computational methods for solving partial differential equations (PDEs) [2]. In this paper we establish the existence, uniqueness, and stability of the stationary solution of the one-dimensional viscous Burgers equation (1) on a finite interval with the Dirichlet boundary conditions (4). We use the Cole-Hopf transformation to give the result for any combination of A and B in the boundary conditions (4). Using a different method (linearization) H.-O. Kreiss and G. Kreiss (1985) gave a similar result for a subset of cases: $A \geq |B|,\ B \leq 0 < A$, as well as for Burgers' equation with forcing [8]. We obtain explicit formulas for solutions and analytically determine the Lyapunov exponents characterizing the asymptotic behavior of arbitrary solutions approaching the stationary solution with the same boundary conditions (4).

1 Explicit formulas for stationary solutions

The viscous Burgers equation is the nonlinear partial differential equation

$$u_t + uu_x = \nu u_{xx} \tag{1}$$

with $\nu = \text{const} > 0$. If we set u_t to zero, for the stationary solution $u = u^S(x)$ we obtain

$$uu_x = \nu u_{xx}. \tag{2}$$

We note that $uu_x = \dfrac{1}{2}(u^2)'_x$, therefore (2) gives

$$2\nu u_x = u^2 + C_0. \tag{3}$$

First, assume that $u_x \neq 0$ and C_0 is negative, $C_0 = -a^2 < 0$ (i. e. $2\nu u_x < u^2$). We have $dx = 2\nu du/(u^2 - a^2)$,

$$\frac{ax}{\nu} = \ln\left(C_1 \left|\frac{a-u}{a+u}\right|\right), \qquad \text{where } C_1 = \left|\frac{a+u(0)}{a-u(0)}\right|.$$

If, in addition, $|u| < a$ (i.e. $u_x < 0$), then

$$u = a\,\frac{C_1 - \exp(ax/\nu)}{C_1 + \exp(ax/\nu)} = -2\nu k_0 \tanh(k_0(x - x_0)), \quad \text{where } k_0 = \frac{a}{2\nu}, \quad x_0 = \frac{1}{k_0}\,\mathrm{artanh}\,\frac{u(0)}{2\nu k_0},$$

while if $|u| > a$ (i.e. $0 < 2\nu u_x < u^2$), then

$$u = a\,\frac{C_1 + \exp(ax/\nu)}{C_1 - \exp(ax/\nu)} = -2\nu k_0 \coth(k_0(x - x_0)), \quad \text{where } k_0 = \frac{a}{2\nu}, \quad x_0 = \frac{1}{k_0}\,\mathrm{arcoth}\,\frac{u(0)}{2\nu k_0}.$$

Now assume that C_0 is positive, $C_0 = a^2 > 0$ (i.e. $2\nu u_x > u^2$). Then $dx = 2\nu du/(u^2 + a^2)$,

$$\frac{ax}{2\nu} = \arctan\frac{u}{a} + C_1, \quad \text{where } C_1 = -\arctan\frac{u(0)}{a}; \qquad \text{hence} \quad u = a\tan\left(\frac{ax}{2\nu} - C_1\right);$$

or, equivalently,

$$u = -2\nu k_0 \cot(k_0(x - x_0)), \qquad \text{where } k_0 = \frac{a}{2\nu}, \quad x_0 = \frac{1}{k_0}\,\mathrm{arccot}\,\frac{u(0)}{2\nu k_0}.$$

Finally, if $C_0 = 0$, then $u = -2\nu/(x - x_0)$; if $u_x = 0$, then $u = \mathrm{const}$ and $u^2 = |C_0| = \mathrm{const}$. For convenience, all explicit formulas for stationary solutions are listed together in Table 1 (left column).

Table 1. Stationary solutions u^S of Burgers equation and the corresponding solutions φ^S of the heat equation (6). $H = 2\nu(B - A) - lAB$; $2\nu k_0 = |C_0|^{1/2}$, where C_0 is the constant in (3); $u^S(x) = -2\nu(\ln|\varphi^S(x,t)|)'_x$.

Solution $u^S(x)$ of (1)	Conditions on u, u_x	Conditions on A, B, H	Solution $\varphi^S(x,t)$ of (6)
(a) $-2\nu k_0 \cot(k_0(x - x_0))$	$0 \le u^2 < 2\nu u_x$	$A < B,\ H > 0$	$C\sin(k_0(x - x_0))\exp(-\nu k_0^2 t)$
$2\nu k_0 \tan(k_0(x - x_0^*))$	*(the same conditions and same solution as above)*		$C\cos(k_0(x - x_0^*))\exp(-\nu k_0^2 t)$
(b) $-2\nu/(x - x_0)$	$0 < u^2 = 2\nu u_x$	$A < B,\ H = 0$	$C(x - x_0)$
(c) $-2\nu k_0 \coth(k_0(x - x_0))$	$0 < 2\nu u_x < u^2$	$A < B,\ H < 0$	$C\sinh(k_0(x - x_0))\exp(\nu k_0^2 t)$
(d) $\pm 2\nu k_0 = \mathrm{const}$	$u_x = 0$	$A = B$	$C\exp(\nu k_0^2 t \mp k_0 x)$
(e) $-2\nu k_0 \tanh(k_0(x - x_0))$	$u_x < 0$	$A > B$	$C\cosh(k_0(x - x_0))\exp(\nu k_0^2 t)$

We will now consider Burgers equation (1) with the Dirichlet boundary conditions on the interval $x \in [0, l]$:

$$u(0,t) = A, \qquad u(l,t) = B, \tag{4}$$

where A and B are constants. Let us find out which explicit formulas (Table 1) can represent the stationary solution u^S of equation (1) with boundary conditions (4). Here we are concerned exclusively with solutions that are continuous, bounded, and sufficiently smooth everywhere on the interval $x \in [0, l]$.

Clearly, when $A > B$, the stationary solution u^S can only have form (e) $-2\nu k_0 \tanh k_0(x - x_0)$ which is the only decreasing function in the left column of Table 1. When $A = B$, the stationary solution u^S can only have form (d); all other explicit formulas for u^S defined on $[0, l]$ are either strictly decreasing or strictly increasing functions of x.

To examine the stationary solution $u^S(x)$ for the trickiest case, $A < B$, we introduce the quantity

$$H = 2\nu(B - A) - lAB.$$

An elementary calculation shows that u^S has form (b) if and only if $H = 0$, $A < B$. It remains to analyze the situations that yield solutions (a) and (c). We note that, at any given point $(x, u(x))$, any graph u^S of form (a) is steeper than (b), while any graph of form (c) is less steep than (b). Indeed, for any stationary solution u^S we have a constant value of $C_0 = 2\nu u_x - u^2$; solutions (a) are obtained from (3) when $2\nu u_x > u^2$ (steeper graphs, $C_0 > 0$, $H > 0$), while solutions (c) are obtained from (3) when $2\nu u_x < u^2$ (less steep graphs, $C_0 < 0$, $H < 0$). Thus when $A < B$ and $H > 0$, we can only have u^S given by formula (a); when $A < B$ and $H < 0$ we can only have u^S given by formula (c).

Note also that we have not yet proved that a stationary solution satisfying boundary conditions (4) *exists* for an arbitrary combination of A and B. (We will prove this in Section 3.) Still, in the simple cases (b) $A < B$, $H = 0$ and (d) $A = B$, it is already obvious that such stationary solutions do exist.

2 The Cole-Hopf transformation

Burgers equation (1) is a rare example of a nonlinear PDE that can be linearized using a simple transformation. Specifically, if in equation (1) we substitute

$$u(x,t) = -2\nu(\ln|\varphi(x,t)|)'_x \tag{5}$$

then for the unknown function $\varphi(x,t)$ we obtain the *heat equation*

$$\varphi_t = \nu\varphi_{xx}. \tag{6}$$

The substitution (5) is known as the Cole-Hopf transformation [1, 2, 5, 6]. Let us discuss some interesting properties of this transformation.

Firstly, transformation (5) can produce the same solution $u(x,t)$ of (1) from many different solutions $\varphi(x,t)$ of (6); these $\varphi(x,t)$ may differ from each other by an arbitrary nonzero mutiplier C. Indeed, $(\ln|\varphi|)'_x = (\ln|C\varphi|)'_x$ for any constant $C \neq 0$.

Secondly, zero values of $\varphi(x,t)$ are mapped by (5) into discontinuities of $u(x,t)$. Therefore, to get a continuous $u(x,t)$, it is not enough to start from a continuous solution $\varphi(x,t)$ of (6). We, moreover, need to restrict ourselves to those solutions $\varphi(x,t)$ that are nonzero everywhere on $[0,l]$ for all $t \geq 0$.

Further, stationary solutions $u^S(x)$ of (1) correspond to solutions $\varphi^S(x,t)$ of (6) that may or may not be stationary. Explicit formulas for those $\varphi^S(x,t)$ that yield stationary solutions $u^S(x)$ are listed in the right column of Table 1. Interestingly, among these $\varphi^S(x,t)$ we find "non-physical" solutions of the heat equation that grow infinitely large when $t \to \infty$.

3 Existence and uniqueness of the stationary solution

Using the Cole-Hopf transformation (5), we will now establish the existence and uniqueness of the stationary solution of (1), (4) for any A and B. Note that (5) transforms the problem (1), (4) into the following problem for heat equation (6) with the Robin boundary conditions:

$$\varphi_t = \nu\varphi_{xx}$$

$$\varphi_x(0,t) + \frac{A}{2\nu}\varphi(0,t) = 0, \qquad \varphi_x(l,t) + \frac{B}{2\nu}\varphi(l,t) = 0. \tag{7}$$

Denote by φ^S the solution of (6) that under transformation (5) yields the stationary solution u^S of (1). Our φ^S must have the form $\varphi^S(x,t) = X(x)\cdot T(t)$. (This can be checked directly by substituting φ^S into (5), or simply by inspection of the right column in Table 1.) Here $X(x)$ is a function of the x coordinate only, and $T(t)$ is a function of time t only. Substituting this φ^S into the heat equation(6) and dividing through by νTX, we get

$$\frac{T'}{\nu T} = \frac{X''}{X} = -\lambda.$$

(One ratio is a function of t only, while the other ratio is a function of x only. In order for these two ratios to be equal, they both must be equal to a constant which we denote $-\lambda$.)

For the function $X(x)$, problem (6), (7) translates into an eigenvalue problem (a Sturm-Liouville problem) with Robin boundary conditions:

$$-X''(x) = \lambda X(x) \tag{8}$$

$$X'(0) + \frac{A}{2\nu}X(0) = 0, \qquad X'(l) + \frac{B}{2\nu}X(l) = 0; \tag{9}$$

and for the function $T(t)$ we readily obtain

$$T(t) = C\exp(-\nu\lambda t). \tag{10}$$

For u^S to be continuous, φ^S must be nonzero everywhere on the interval $[0, l]$. So the question now is: how many eigenfunctions of (8), (9) are nonzero everywhere on $[0, l]$? The answer is well known: for any A and B, there is one and only one such eigenfunction. This follows from the familiar fact that, for any A and B in problem (8), (9), all eigenvalues λ_i ($\lambda_0 < \lambda_1 < \ldots$) have multiplicity 1, and the respective eigenfunction $X_i(x)$ has exactly i zeros inside the interval $(0, l)$; see [3, pp. 14-18]. Thus, in problem (8), (9) we are interested in the eigenfunction $X_0(x)$ that has no zeros for $x \in [0, l]$ and corresponds to the least eigenvalue λ_0. For φ^S we find, up to a nonzero multiplier C,

$$\varphi^S(x,t) = CX_0(x) \cdot \exp(-\nu\lambda_0 t) \qquad (\varphi^S \text{ has no zeros for } x \in [0, l]).$$

Therefore, for any A and B, *there exists a unique stationary solution* $u^S(x)$ of Burgers equation (1) with boundary conditions (4):

$$u^S(x) = -2\nu(\ln|\varphi^S(x,t)|)'_x = -2\nu(\ln|X_0(x)|)'_x.$$

4 Stability and Lyapunov exponents

Now let us study the evolution of the absolute value $|u - u^S|$ for an arbitrary non-stationary solution

$$u(x,t) = -2\nu(\ln|\varphi(x,t)|)'_x = -2\nu\frac{\varphi_x(x,t)}{\varphi(x,t)},$$

where both $u(x,t)$ and $u^S(x)$ satisfy the Burgers equation (1) with boundary conditions (4), and $\varphi(x,t)$ is a suitable positive solution of (6). It is known that the solution $u(x,t)$ exists for "reasonable" combinations of the boundary conditions (4) and initial condition $u(x, 0)$ [9]. We say that u^S is *stable* if $|u - u^S| \to 0$ as $t \to \infty$, for an arbitrary u obeying (1), (4). We have

$$\begin{aligned} |u - u^S| &= 2\nu\left|\frac{\varphi^S_x}{\varphi^S} - \frac{\varphi_x}{\varphi}\right| = 2\nu\left|\frac{\varphi\varphi^S_x - \varphi^S\varphi_x}{\varphi^S\varphi}\right| \\ &= 2\nu\left|\frac{\varphi(\varphi^S_x - \varphi_x) + \varphi_x(\varphi - \varphi^S)}{\varphi^S\varphi}\right| = 2\nu\left|\frac{\varphi_x\tilde{\varphi} - \varphi\tilde{\varphi}_x}{\varphi^S\varphi}\right|. \end{aligned}$$

Here we have introduced the notation $\tilde{\varphi} = \varphi - \varphi^S$. Taking into account that $u = -2\nu\varphi_x/\varphi$, for all $x \in [0, l]$ and all $t \geq 0$ we obtain the estimate

$$|u - u^S| \leq |u| \cdot \left|\frac{\tilde{\varphi}}{\varphi^S}\right| + 2\nu\left|\frac{\tilde{\varphi}_x}{\varphi^S}\right| \leq \max_{x\in[0,l]}|u(x,0)| \cdot \left|\frac{\tilde{\varphi}}{\varphi^S}\right| + 2\nu\left|\frac{\tilde{\varphi}_x}{\varphi^S}\right|. \qquad (11)$$

In inequality (11) we have used the maximum principle for Burgers equation: the solution $u(x,t)$ attains its maximum either in the initial value $u(x, 0)$ or at the boundary of the interval $[0, l]$. (A discussion of maximum principles for PDEs can be found in [4, 7, 9]. The proof of the maximum principle for Burgers equation is similar to that for linear parabolic PDEs.)

Expand $\varphi(x,t)$ in a series over the system of eigenfunctions $X_i(x)$ of (8), (9):

$$\varphi(x,t) = \sum_{i=0}^{\infty}\alpha_i X_i(x)T_i(t) = \sum_{i=0}^{\infty}\varphi_i(x,t), \quad T_i(t) = \exp(-\nu\lambda_i t), \quad T_i(0) = 1. \qquad (12)$$

In this series, the term $\varphi_0(x,t) = \alpha_0 X_0(x)T_0(t)$ is the same as φ^S (Table 1) up to a constant nonzero multiplier. Let us choose C in the expression of φ^S (Table 1) so that $\varphi_0 = \varphi^S$. If we now compute the difference $\varphi - \varphi^S$, the term $\varphi_0(x,t)$ will cancel out, and we get

$$\tilde{\varphi} = \varphi - \varphi^S = \sum_{i=1}^{\infty}\varphi_i(x,t). \qquad (13)$$

Since $T_i(t) = \exp(-\nu\lambda_i t)$, we see that $\varphi_1(x,t)$ becomes the *largest* term in (13) when $t \to \infty$ (assuming $\alpha_1 \neq 0$ in (12)). We then have

$$\max_{x\in[0,l]}|\varphi^S| \asymp \exp(-\nu\lambda_0 t), \quad \max_{x\in[0,l]}|\tilde{\varphi}| \asymp \exp(-\nu\lambda_1 t), \quad \max_{x\in[0,l]}|\tilde{\varphi}_x| \asymp \exp(-\nu\lambda_1 t) \qquad \text{as } t \to \infty,$$

so the estimate (11) results in

$$\max_{x\in[0,l]} |u - u^S| \asymp \exp(-\nu(\lambda_1 - \lambda_0)t) \qquad \text{as } t \to \infty. \tag{14}$$

This paves the way to proving the *stability of the stationary solution* u^S. Indeed, the difference $|u - u^S|$ is an exponentially vanishing quantity when $t \to \infty$. Nevertheless, the convergence of $|u - u^S|$ to zero might turn out to be very slow; this is the case when the least two eigenvalues λ_0 and λ_1 in problem (8), (9) differ only slightly.

We got the estimate (14) under the assumption that $\alpha_1 \neq 0$ in (12), that is, in the series expansion of φ over the system of eigenfunctions $X_i(x)$ there is a nonzero term φ_1 containing the eigenfunction $X_1(x)$. However, if it so happens that one or more initial terms in (13) are zero, then the series (13) for $\tilde{\varphi} = \varphi - \varphi^S$ will start at some φ_n $(n > 1)$. In the general case, therefore, instead of (14) we would have

$$\max_{x\in[0,l]} |u - u^S| \asymp \exp(-\nu(\lambda_n - \lambda_0)t) \qquad \text{as } t \to \infty, \tag{15}$$

where n is the number of the first nonzero term in the series expansion of $\tilde{\varphi} = \varphi - \varphi^S$ (13). We have thus proved that the stationary solution u^S is stable: $|u - u^S| \to 0$ as $t \to \infty$.

Note that the functions φ_i $(i = 1, 2, \ldots)$ in (12) have the same explicit formulas as φ^S (Table 1), except that each φ_i contains its own values in place of k_0 and x_0; let us denote these new constant values by k_i and x_i, respectively.

All constants k_i and x_i can be found if we substitute the general solutions of (8) (trigonometric, exponential or hyperbolic functions) for the eigenfunctions $X_i(x)$ $(i = 0, 1, 2, \ldots)$ in the boundary conditions (9). In most cases (i. e., cases (a), (c), (e) in Table 1), this substitution yields the following transcendent equations for $\xi_i = k_i l$:

$$\cot \xi_i = \frac{p}{\xi_i} + q\xi_i \quad \text{for } \varphi_i \text{ of form (a) in Table 1}, \quad \lambda_i = k_i^2 > 0, \quad \text{and} \tag{16}$$

$$\coth \xi_i = \frac{p}{\xi_i} - q\xi_i \quad \text{for } \varphi_i \text{ of form (c) or (e) in Table 1}, \quad \lambda_i = -k_i^2 < 0, \tag{17}$$

$$\text{where} \quad \xi_i = k_i l > 0, \quad p = \frac{lAB}{2\nu(B-A)}, \quad q = \frac{2\nu}{l(B-A)}.$$

The transcendent equation (16), with $\cot \xi_i$, may correspond to *any* i, whereas equation (17), with $\coth \xi_i$, may correspond only to $i = 0, 1$ (the least two eigenvalues λ_0, λ_1) because hyperbolic functions cannot have more than one zero value on the interval $[0, l]$.

When $A = B$ in (4) and (9), we have an exceptional case: all k_i and λ_i can be found in a closed form. Here the interval $[0, l]$ contains a whole number of semiperiods of the eigenfunction $X_i(x) = \sin(k_i(x - x_i))$, $i = 1, 2, \ldots$, which readily yields

$$k_i = \frac{\pi i}{l} \ (i = 1, 2, \ldots), \quad \text{while} \quad k_0 = \frac{|A|}{2\nu}, \quad X_0(x) = C \exp(\pm k_0 x); \quad \text{see Table 1 (d).}$$

Therefore, if $A = B$, we find

$$\lambda_n = \left(\frac{\pi n}{l}\right)^2 \ (n \geq 1), \quad \lambda_0 = -\left(\frac{A}{2\nu}\right)^2, \quad \text{and} \quad \lambda_n - \lambda_0 = \left(\frac{\pi n}{l}\right)^2 + \left(\frac{A}{2\nu}\right)^2; \quad \text{cf. (14), (15).}$$

Now we will reuse the customary definition of *Lyapunov exponents* in the context of problem (1), (4) for Burgers equation. Let $u(x, t)$ be a solution of (1),(4). The Lyapunov exponent μ of this solution is defined as

$$\mu = \limsup_{t\to\infty} \frac{\ln \|u - u^S\|}{t}. \tag{18}$$

This definition, in general, depends on our choice of the norm $\|\cdot\|$. If $u(x, t)$ behaves so that $\|u - u^S\| \asymp \exp(\delta t)$ as $t \to \infty$, then it is easy to see that δ is the Lyapunov exponent of this $u(x, t)$.

Let us use the norm defined as the maximum absolute value:

$$\|w(x)\| = \max_{x\in[0,l]} |w(x)|.$$

Then estimates (14), (15) allow us to determine all Lyapunov exponents for any $u(x,t)$ satisfying (1), (4):

$$\mu_i = -\nu(\lambda_i - \lambda_0), \quad i = 1, 2, \ldots, \tag{19}$$

where, as before, λ_i are eigenvalues of (8), (9). Solutions $u(x,t)$ corresponding to the Lyapunov exponents μ_i can be written simply as

$$u_i(x,t) = -2\nu(\ln|\varphi^S(x,t) + \varphi_i(x,t)|)'_x, \quad i = 1, 2, \ldots,$$

where $\varphi_i(x,t)$ is the respective term of (12). For example, when u^S has the form (a) in Table 1, we have

$$\begin{aligned} \varphi^S(x,t) &= C\sin(k_0(x-x_0))\exp(-\nu k_0^2 t) && (\varphi^S \text{ has no zeros for } x \in [0,l]), \\ \varphi_i(x,t) &= \alpha_i \sin(k_i(x-x_i))\exp(-\nu k_i^2 t) && (\varphi_i \text{ has } i \text{ zeros for } x \in [0,l]), \end{aligned}$$

and we can write a solution $u_i(x,t)$ corresponding to the Lyapunov exponent μ_i as follows:

$$u_i(x,t) = -2\nu \frac{Ck_0\cos(k_0(x-x_0)) + \alpha_i k_i \cos(k_i(x-x_i)) \cdot \exp(-\nu(k_i^2 - k_0^2)t)}{C\sin(k_0(x-x_0)) + \alpha_i \sin(k_i(x-x_i)) \cdot \exp(-\nu(k_i^2 - k_0^2)t)}. \tag{20}$$

Because each individual term in series (12) satisfies the Robin boundary conditions (7), each function $u_i(x,t)$ defined as above must satisfy the Dirichlet boundary conditions (4).

We have thus determined the Lyapunov exponents in the nonlinear problem (1), (4) for Burgers equation: we have found that formula (19) relates the Lyapunov exponents μ_i to the eigenvalues λ_i of the linear problem (8), (9). All Lyapunov exponents μ_i are negative; there are countably many of them; we can write explicit formulas for the corresponding solutions $u_i(x,t)$ of Burgers equation (1). This is an interesting example of a situation where one can analytically determine the Lyapunov exponents for solutions of a nonlinear PDE with Dirichlet boundary conditions.

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