

On the geometric mean of the first n primes

Alexei Kourbatov
 www.JavaScripter.net/math
 akourbatov@gmail.com

Abstract

Let p_n be the n th prime, and consider the sequence $s_n = (2 \cdot 3 \cdots p_n)^{1/n} = (p_n\#)^{1/n}$, the geometric mean of the first n primes. We give a short proof that $p_n/s_n \rightarrow e$, a result conjectured by Vrba (2010) and proved by Sándor & Verroken (2011). We show that $p_n/s_n = \exp(1 + 1/\log p_n + O(1/\log^2 p_n))$ as $n \rightarrow \infty$, and give explicit lower and upper bounds for the $O(1/\log^2 p_n)$ term.

1 Introduction

In 2001 A. Murthy posted OEIS sequence [A062049](#): the integer part of the geometric mean of the first n primes [8]. The sequence is non-decreasing, unbounded, and begins as follows:

2, 2, 3, 3, 4, 5, 6, 7, 8, 9, 10, 11, 13, 14, 15, 16, 17, 19, 20, 21, 23 . . .

Let p_n be the n th prime, and let s_n denote the geometric mean of the first n primes,

$$s_n = (2 \cdot 3 \cdots p_n)^{1/n} = (p_n\#)^{1/n}, \quad \text{where} \quad p_n\# = 2 \cdot 3 \cdots p_n = \prod_{k=1}^n p_k,$$

then $\text{A062049}(n) = \lfloor s_n \rfloor$. (The product $p_n\#$ is called the primorial of p_n ; see [A002110](#).)

For many years, sequence [A062049](#) has been lacking an asymptotic formula; nor did it have any lower or upper bounds for the sequence terms. In 2010 A. Vrba conjectured [5] that

$$p_n/s_n \rightarrow e \quad \text{as } n \rightarrow \infty.$$

This was proved in 2011 by Sándor and Verroken [7], and revisited in 2013 by Hassani [3].

In Section 2 we give a new short proof that $p_n/s_n \rightarrow e$ and, moreover, we show that

$$p_n/s_n = \exp(1 + 1/\log p_n + O(1/\log^2 p_n)).$$

We give explicit lower and upper bounds for the $O(1/\log^2 p_n)$ term.

2 The main result

Let $\pi(x)$ denote the prime counting function and $\theta(x)$ denote Chebyshev's θ function:

$$\begin{aligned}\pi(x) &= \sum_{\substack{p \leq x \\ p \text{ prime}}} 1; \\ \theta(x) &= \sum_{\substack{p \leq x \\ p \text{ prime}}} \log p = \log \prod_{\substack{p \leq x \\ p \text{ prime}}} p.\end{aligned}$$

Clearly $\pi(p_n) = n$ and $\theta(p_n) = \log(p_n\#)$, so $\log s_n = \log(p_n\#)/n = \theta(p_n)/\pi(p_n)$.

Lemma 1. *For $x \geq 10^8$ we have*

$$\frac{|\theta(x) - x|}{\pi(x)} < \frac{1}{\log^2 x}.$$

Proof. Let $x \geq 10^8$. From Dusart [2] we have the inequalities

$$\begin{aligned}|\theta(x) - x| &< \frac{x}{\log^3 x} \quad \text{for } x \geq 89967803 \quad [2, \text{Theorem 5.2}], \\ \pi(x) &> \frac{x}{\log x - 1} \quad \text{for } x \geq 5393 \quad [2, \text{Theorem 6.9}].\end{aligned}$$

Combining the above inequalities we get

$$\frac{|\theta(x) - x|}{\pi(x)} < \frac{x}{\log^3 x} \cdot \frac{\log x - 1}{x} < \frac{1}{\log^2 x}$$

for all $x \geq 10^8$, as desired. □

Theorem 2. *If $s_n = (p_n\#)^{1/n}$, then $p_n/s_n \rightarrow e$ as $n \rightarrow \infty$, and for $p_n \geq 32059$ we have*

$$\exp\left(1 + \frac{1}{\log p_n} + \frac{1.62}{\log^2 p_n}\right) < p_n/s_n < \exp\left(1 + \frac{1}{\log p_n} + \frac{4.83}{\log^2 p_n}\right). \quad (1)$$

Proof. Let $x \geq 10^8$. From Axler [1, Corollaries 3.5, 3.6] we have

$$\log x - 1 - \frac{1}{\log x} - \frac{3.83}{\log^2 x} < \frac{x}{\pi(x)} < \log x - 1 - \frac{1}{\log x} - \frac{2.62}{\log^2 x}.$$

Therefore,

$$1 + \frac{1}{\log x} + \frac{2.62}{\log^2 x} < \log x - \frac{x}{\pi(x)} < 1 + \frac{1}{\log x} + \frac{3.83}{\log^2 x}, \quad (2)$$

while

$$\log x - \frac{x}{\pi(x)} - \frac{|\theta(x) - x|}{\pi(x)} < \log x - \frac{\theta(x)}{\pi(x)} < \log x - \frac{x}{\pi(x)} + \frac{|\theta(x) - x|}{\pi(x)}. \quad (3)$$

Combining (2) and (3) with the bound $\frac{|\theta(x)-x|}{\pi(x)} < \frac{1}{\log^2 x}$ (Lemma 1), for $x \geq 10^8$ we get

$$1 + \frac{1}{\log x} + \frac{1.62}{\log^2 x} < \log x - \frac{\theta(x)}{\pi(x)} < 1 + \frac{1}{\log x} + \frac{4.83}{\log^2 x}. \quad (4)$$

But $\log(p_n/s_n) = \log p_n - \theta(p_n)/\pi(p_n)$, so setting in (4) $x = p_n > 10^8$ we find

$$1 + \frac{1}{\log p_n} + \frac{1.62}{\log^2 p_n} < \log(p_n/s_n) < 1 + \frac{1}{\log p_n} + \frac{4.83}{\log^2 p_n}. \quad (5)$$

Exponentiating (5) we prove the theorem for $p_n > 10^8$. Separately, we verify by computation that (1) is true for $32059 \leq p_n < 10^8$ as well. \square

Remarks.

(i) The convergence $p_n/s_n \rightarrow e$ is slow (see Table 1). The better approximation

$$p_n/s_n \approx \exp\left(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n}\right) \quad (6)$$

has a relative error well below 1% for $p_n > 10^6$, even while p_n/s_n is still far from e .

(ii) One can construct approximations with more¹ terms:

$$p_n/s_n \approx \exp\left(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n} + \frac{13}{\log^3 p_n} + \dots\right),$$

where the coefficients 1, 3, 13, ... are terms of OEIS sequence [A233824](#): a recurrent sequence in Panaitopol's formula for $\pi(x)$ [4]. A rigorous proof of such approximations, akin to Theorem 2, would depend on sharper bounds for $\frac{x}{\pi(x)}$ and $\frac{|\theta(x)-x|}{\pi(x)}$, and these sharper bounds may in turn depend, e. g., on the truth of the Riemann Hypothesis.

Table 1: Values of n , p_n , $s_n = (p_n\#)^{1/n}$, p_n/s_n and approximation (6) for $p_n \approx 10^k$

n	p_n	s_n	p_n/s_n	$\exp(1 + \frac{1}{\log p_n} + \frac{3}{\log^2 p_n})$
5	11	4.706764	2.337062	6.950270
26	101	29.899069	3.378032	3.886576
169	1009	298.623420	3.378837	3.344393
1230	10007	3143.242209	3.183655	3.139064
9593	100003	32619.709536	3.065723	3.032817
78499	1000003	334329.282286	2.991072	2.968628
664580	10000019	3401979.209240	2.939471	2.925864
5761456	100000007	34435454.560637	2.903984	2.895414
50847535	1000000007	347413774.453987	2.878412	2.872666

¹ The number of terms is meant to be finite, while p_n should be large enough; otherwise, such approximations would actually be worse than those with fewer terms. When p_n is small, even approximation (6) itself is worse than $p_n/s_n \approx \exp(1 + \frac{1}{\log p_n})$ or $p_n/s_n \approx e$ (see, e. g., the first line in Table 1, $p_n = 11$).

(iii) Bounds (1) strengthen the double inequality of Sándor [6]

$$e < p_n/s_n < \frac{p_n}{p_{n-1}} \cdot p_{n+1}^{\pi(n)/n} \quad \text{for } n \geq 10.$$

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(Concerned with sequences [A002110](http://oeis.org/A002110), [A062049](http://oeis.org/A062049), [A233824](http://oeis.org/A233824).)
